International Journal of Theoretical Physics, Vol. 24, No. 9, 1985

Dirac's Large Numbers Hypothesis in Einstein's Theory of Gravitation

Ronald Gautreau[1]

Received March 5, 1985

A method is given for incorporating Dirac's Large Numbers hypothesis into Einstein's standard theory of general relativity. The method involves the assumption that at each point in space there exist two types of clocks, a cosmological clock measuring ephemeris time τ and an atomic clock measuring atomic time t_A. Newton's law of universal gravitation is formulated relativistically in terms of these two times and the proper distance determined by measuring rods between simultaneous events, and a method is given for operationally identifying G. The Large Numbers hypothesis requirement that $G_A \propto 1/t_A$ is then used to establish the relationship between the two times. Alternative derivations of the time relationship not involving a time-varying gravitational "constant" are obtained by intercomparison of various large numbers. It is shown that the resulting relationship between t_A and τ gives agreement with the observed natural microwave radiations. Also, the Large Numbers hypothesis leads to a time formed from the fundamental constants of Nature that is comparable to the age of the Universe.

1. INTRODUCTION

Building on an observation put forth by Eddington (1931), Dirac (1938) has pointed out that there are certain large dimensionless numbers (LN's) that can be constructed from the physical parameters of Nature, such as

$$\mathrm{LN1} = \frac{\text{Coulomb force between proton and electron}}{\text{gravitational force between proton and electron}}$$

$$= \frac{ke^2}{Gm_e m_p} = 2.3 \times 10^{39} \tag{1}$$

[1]Physics Department, New Jersey Institute of Technology, Newark, New Jersey 07102.

0020-7748/85/0900-0877$04.50/0

$$\text{LN2} = \frac{\text{the age of the Universe}}{\text{time for light to traverse an electron}}$$

$$= \frac{t}{ke^2/m_e c^3} \approx 10^{39} \tag{2}$$

$$(\text{LN3})^2 = \frac{\text{mass of Universe moving away from us with speed} < c/2}{\text{mass of a proton}}$$

$$\approx (10^{39})^2 \tag{3}$$

Dirac believed that it must be more than coincidence that these large dimensionless numbers are so closely related. He has therefore proposed a Large Numbers hypothesis (LNh) that asserts that large dimensionless numbers in Nature must be interrelated by equations where the coefficients are close to unity.

Since the large number (2) varies with the age of the Universe, it then follows from the LNh that the other large numbers must also vary with the age of the Universe. Combining (2) with (1), and assuming that the variable quantity is G, Dirac gets

$$G \propto t^{-1} \tag{4}$$

Similarly, combining (2) with (3), Dirac gets

$$(\text{LN3})^2 \propto t^2 \tag{5}$$

The LNh arguments given here are based upon Newtonian ideas, with t being the absolute Newtonian time. There remains the problem of incorporating the LNh concept into the formalism of general relativity and quantum theory. One difficulty encountered is that the Newtonian parameter G appears as a constant on the right-hand side of Einstein's field equations $G^{\rho\sigma} = -(8\pi G/c^4)T^{\rho\sigma}$ because of the requirement that the field equations must give the appropriate Newtonian limit where spacetime becomes flat. If G (or c) were to vary with time, the equality would be upset.

To get around this difficulty, Dirac (1979) has assumed that there are two different systems of units of length and time in Nature, an Einstein (E) system and an atomic (A) system. In the E system, time is measured by the ephemeris time governing the motions of galaxies and planets, while in the A system time is measured by the periodic vibrations of atoms. Dirac required that there must be two different metrics for the Universe, an Einstein metric ds_E and an atomic metric ds_A. From purely dimensional arguments, Dirac worked out that the two metrics must be related by

$$ds_E = t_A\, ds_A \tag{6}$$

from which he determined that the ephemeris time τ and the atomic time t_A are related by

$$\tau = \tfrac{1}{2}(t_A^2/t_{An}) \qquad (t_{An} = \text{present epoch}) \tag{7}$$

Working in the A system of units and using dimensional arguments with (5), Dirac determined that the galaxies will evolve according to

$$R_A = kt_A^{1/3}, \qquad k = \text{const} \tag{8}$$

Dirac finally concluded that the model of the Universe that will be in agreement with his LNh and the E and A metrics is the Einstein-de Sitter (ES) Universe, which is a zero curvature Universe where the cosmological fluid has zero pressure and where the cosmological constant equals zero. We will discuss the ES Universe in more detail in Section 3.

In this paper we develop an approach for incorporating the LNh into general relativity that is different from the way Dirac proceeded. We shall adopt the basic idea from the LNh that cosmological processes may evolve at rates that do not remain synchronous with the evolution of atomic processes. This means that if one has a cosmological clock coincident with an atomic clock, the ratio of the periods of the two clocks will vary with the age of the Universe. This fact by itself in no way requires the need for two metrics nor two different length standards. It simply means that along the world line of a coincident cosmological clock and atomic clock there will be two different rates of "ticking" of clocks, a cosmological rate $d\tau$ corresponding to the evolution of galaxies and an atomic rate dt_A corresponding to the evolution of atoms, and the ratio $d\tau/dt_A$ of the two types of tickings need not remain constant as the Universe evolves.

If we possessed a complete theory of atomic processes combined with gravitational processes, the ratio $d\tau/dt_A$ could be worked out from first principles. However, such a unified theory does not exist at present. To make progress, we must therefore resort to some additional principles or hypotheses to guide our thinking. As will be seen, the LNh will be the vehicle that will guide us to the relationship between t_A and τ.

Our approach is founded on previous work we have done (Gautreau, 1984) in reformulating cosmological theory in terms of coordinates (R, τ), where τ is the ephemeris time measured by cosmological processes and the curvature coordinate R is the proper distance determined by measuring rods between events that are simultaneous relative to the ephemeris time τ. This is discussed in Sections 2 and 3. The main idea behind our approach is to have a cosmological clock and an atomic clock located together at the same point, and thereby moving along the same common geodesic world line through space-time, as described in Section 4. The relationship between τ and t_A is obtained by using the LNh together with an operational way of identifying G from Newton's law of universal gravitational attraction, as discussed in Sections 5-7. In Section 9 we show how our approach is in agreement with observations of the natural microwave radiation. Alternative derivations of our resulting $\tau - t_A$ relationship based upon large numbers that do not involve a time-varying gravitational constant are given in Section 11. The time variation of the various LN's are summarized in Section 12, and Section 13 shows how the age of the Universe can be related to fundamental constants of Nature.

2. REFORMULATION OF COSMOLOGICAL THEORY

Standard discussions of zero curvature cosmologies usually start with the metric for the Universe expressed in terms of isotropic coordinates (r, τ) as

$$ds^2(r, \tau) = e^{2h(\tau)}(dr^2 + r^2\, d\Omega^2) - c^2\, d\tau^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{9}$$

A particular Universe is determined by specifying the expansion parameter $e^{h(\tau)}$. The metric is "isotropic" in form because its spatial part is proportional to $dr^2 + r^2\, d\Omega^2$.

We have recently shown (Gautreau, 1984) that it is possible to develop a consistent theory of cosmology that does not begin with the isotropic metric form (9), but rather starts by expressing the metric for the Universe in terms of curvature coordinates (R, T) in the form

$$ds^2(R, T) = A^{-1}(R, T)\, dR^2 + R^2\, d\Omega^2 - B(R, T)c^2\, dT^2 \tag{10}$$

The metric (10) is "curvature" in form because the angular part is equal to $R^2\, d\Omega^2$. One then replaces the curvature time coordinate T with the ephemeris time τ measured by clocks in the cosmological fluid (the galaxies) to get the metric in terms of measured coordinates as

$$ds^2(R, \tau) = [dR - RH(\tau)\, d\tau]^2 + R^2\, d\Omega^2 - c^2\, d\tau^2, \qquad H(\tau) = dh/d\tau \tag{11}$$

The coordinates (R, τ) are referred to as "measured" for the following reason: If one looks at the subspace $\tau = \text{const}$, one finds from (11)

$$ds^2(R, \tau = \text{const}) = dR^2 + R^2\, d\Omega^2 \tag{12}$$

from which one finds that the proper distance ΔL in this subspace between two events lying along the same radius is

$$\Delta L = \int_{\tau = \text{const}} ds = \Delta R \tag{13}$$

This means that the curvature spatial coordinate R has the physical significance that it measures proper distance between τ-simultaneous events. We have previously pointed out this interpretation for R in the Schwarzschild field (Gautreau and Hoffmann, 1978). Thus, in the (R, τ) coordinate system of the metric form (11) one has the picture where τ is the ephemeris time measured by the motions of planets and galaxies, while R is the proper distance that is determined by measuring rods between events at any given cosmological time $\tau = \text{const}$.

If desired, one can transform (11) into (9) by changing the spatial coordinate from R to r via

$$R = r\, e^{h(\tau)} \tag{14}$$

But with the approach taken here, this will not be necessary.

3. THE EINSTEIN–DE SITTER UNIVERSE

From now on we will consider only an expanding Einstein–de Sitter (ES) Universe, which has zero pressure and zero curvature with the cosmological constant equal to zero. Dirac (1979) claims the ES Universe is in agreement with his LNh. The ES Universe in each of the three coordinate systems mentioned in Section 2 appears as follows (Gautreau, 1984). In (R, τ) coordinates

$$ds^2(R, \tau) = [dR - (2R/3c\tau)c\, d\tau]^2 + R^2\, d\Omega^2 - c^2\, d\tau^2 \tag{15}$$

The time transformation that takes one from ephemeris time τ to curvature time T is

$$T = \tau[1 + \tfrac{1}{2}(2R/3c\tau)^2]^{3/2} \tag{16}$$

giving the ES metric in (R, T) coordinates as

$$ds^2(R, T) = \frac{dR^2}{1 - (2R/3c\tau)^2} + R^2\, d\Omega^2 - c^2 \frac{dT^2}{[1 - (2R/3c\tau)^2][1 + \frac{1}{2}(2R/3c\tau)^2]} \tag{17}$$

in which $\tau(R, T)$ is given implicitly by (16). The expression for the metric for the ES Universe in (r, τ) coordinates is obtained by changing from the spatial coordinate R used in (15) to a new comoving spatial coordinate r defined by

$$R = r(\tau/\tau_n)^{2/3} \tag{18}$$

to get the well-known isotropic form

$$ds^2(r, \tau) = (\tau/\tau_n)^{4/3}(dr^2 + r^2\, d\Omega^2) - c^2\, d\tau^2 \tag{19}$$

The following discussions will be centered mainly around the form (15), whose subspace $\tau = \text{const}$ is given by (12), so, as described in Section 2, R is the proper distance between τ-simultaneous events.

The measured coordinates (R, τ) give a new way of viewing the evolution of the ES Universe after the big bang at $\tau = 0$. When one works out the null and timelike geodesics for the metric form (15), one gets the trajectories shown on the (R, τ) space-time diagram of Figure 1. The trajectories of the geodesically moving galaxies in Figure 1 are given by

$$R = b\tau^{2/3}, \qquad b = \text{const} \tag{20}$$

The constant b is a measure of the energy that a galaxy had at the big bang, measured relative to our galaxy at the origin $R = 0$. The usual picture in terms of the isotropic coordinates (r, τ) used in (19) is shown in Figure 2, which repeats the world lines of the (R, τ) space-time diagram for Figure 1. Since r is a comoving spatial coordinate, the trajectories of the geodesic galaxies are vertical lines in Figure 2. In contrast, in terms of the measured proper distance R, Figure 1 shows all the galaxies "exploding" with infinite speeds from $R = 0$ at the big bang at $\tau = 0$, and then evolving along geodesic trajectories given by (20). There is nothing intrinsically different between Figures 1 and 2; each figure is simply a coordinate deformation of the other. These and other features of the ES Universe are discussed in detail in Gautreau (1984).

4. TIMES IN THE ES UNIVERSE

The evolution of cosmological processes is determined by the Einstein field equations of general relativity, and proceeds according to ephemeris time τ. One can picture this in Figure 1 (or Figure 2) by looking at the world line of a typical galaxy, say *aeb*. As the galaxy *aeb* evolves from the big bang at $\tau = 0$, the evolution of the stars, planets, etc., in the galaxy are determined by the ephemeris time τ plotted on the ordinate of the (R, τ) space-time diagram of Figure 1 [or the ordinate of the (r, τ) space-time diagram of Figure 2].

In addition to evolution of stars, there will also be evolution of atoms in a galaxy such as *aeb*. Atomic evolution is governed by atomic theory, which, at our present state of knowledge of physics, is independent of general relativity. As such, atomic time t_A according to which atomic processes evolve is independent of the ephemeris time τ discussed above.

It is usually assumed that the evolution of stars and the evolution of atoms in a galaxy proceed synchronously, i.e., that ephemeris time τ and atomic time t_A are equal to each other. However, because general relativity and atomic theory are two separate, independent theories, this usual *assumption* need not necessarily be correct. In fact, the LNh suggests otherwise.

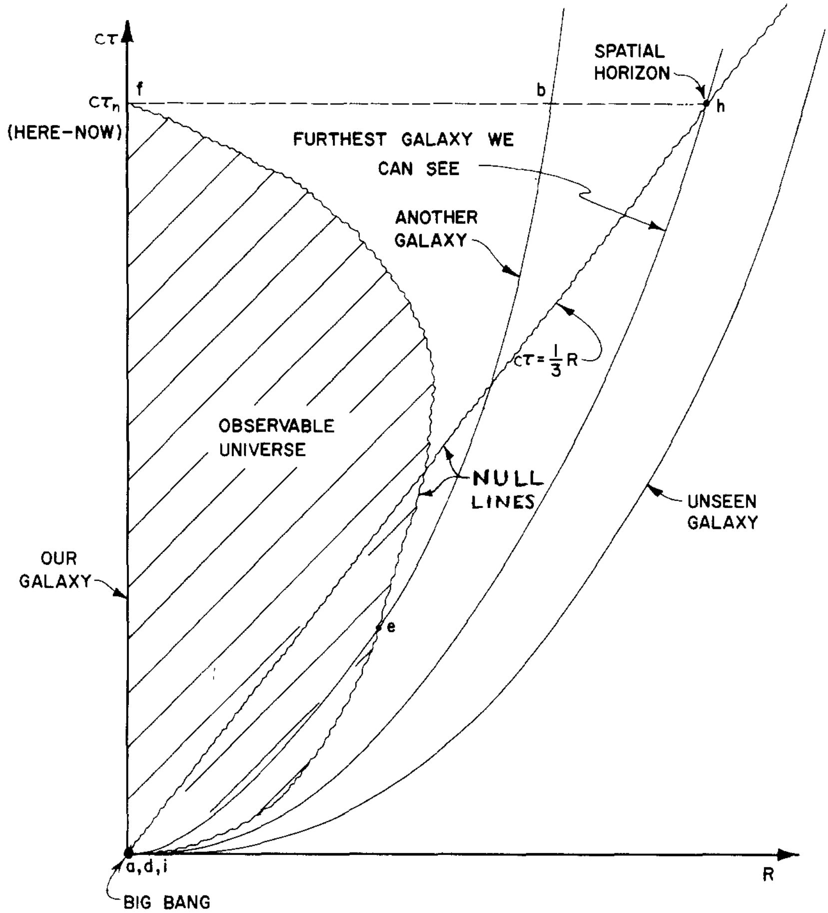

Fig. 1. An (R, τ) space-time diagram of galaxies and light signals in the Einstein–de Sitter (ES) Universe. The galaxies "explode" from $R=0$ at the big bang at $\tau=0$, and follow geodesic trajectories given by $R=b\tau^{2/3}$. The galaxy *dh* is the furthest galaxy we can see, in that the light from this galaxy reaching us now at our present epoch $\tau=\tau_n$ was emitted at the big bang at $\tau=0$. The light from a galaxy further out such as *ij* will not reach us until some time after τ_n, and so is not yet visible to us. The present distance *fh* to the galaxy *dh*, which corresponds to the intersection of *dh* with the null line $c\tau=R/3$, defines the location of the spatial horizon at our present epoch $\tau=\tau_n$. The cross-hatched region denotes the observable Universe, i.e., all the galaxies that can presently be seen by us. For example, the light from galaxy *aeb* that we see now was emitted from *aeb* at event *e*.

If τ and t_A are not equal to each other, then along the geodesic world line of a galaxy such as *aeb* there will be some functional relationship between τ and t_A:

$$\tau=f(t_A) \tag{21}$$

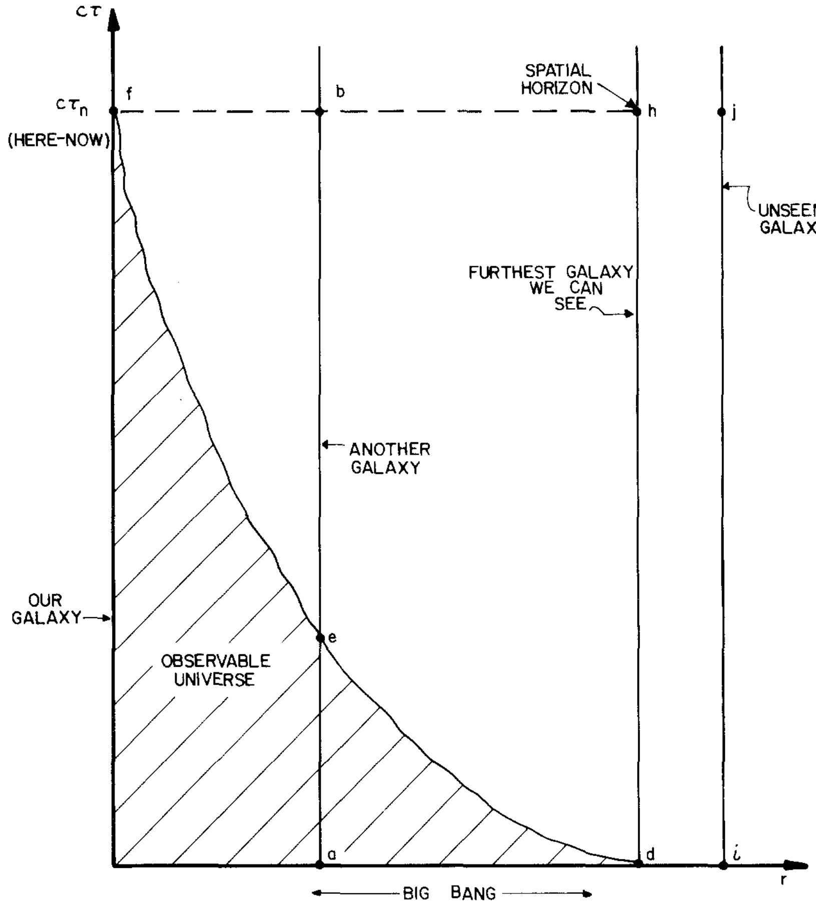

Fig. 2. Repeating Figure 1 in isotropic coordinates (r, τ). Since the spatial coordinate r is a comoving coordinate, the trajectories of galaxies are given by the vertical lines $r = \text{const}$. This figure is simply a coordinate deformation of Figure 1.

Presumably this relationship will be monotonic. If we now invoke the cosmological principle that there is nothing that distinguishes one geodesic galaxy from any other geodesic galaxy, the functional relationship (21) will be the same for *all* galaxies. This means in essence that $\tau = f(t_A)$ in (21) defines a time coordinate transformation independent of the spatial coordinates.

It is important to keep in mind that both of the times τ and t_A in (21) have direct physical significance, and are not simply mathematical constructs. The time τ measures the "ticks" recorded by cosmological clocks, while the time t_A measures the "ticks" recorded by atomic clocks. These ticks are, in principle, numbers that can be counted and recorded. As such, the ticks τ and t_A are independent of any coordinates that we might choose to use for describing the evolution of the Universe.

We can, if we desire, change from describing the evolution of cosmological processes by means of clocks recording ephemeris time τ, and instead use atomic clocks that record atomic time t_A. This change is equivalent to making a time coordinate transformation from τ to t_A according to the (as yet undetermined) functional relationship (21). Even though the atomic clocks may not be "good" clocks for describing cosmological processes, there is no reason in principle why we cannot use them for time measurements.

To express the ES metric in (R, t_A) coordinates, we obtain from (21)

$$d\tau = f'\, dt_A, \qquad f' = df/dt_A \tag{22}$$

which when substituted into the ES metric form (15) yields

$$ds^2(R, t_A) = [dR - (2R/3cf)f'c\, dt_A]^2 + R^2\, d\Omega^2 - (f')^2 c^2\, dt_A^2 \tag{23}$$

With τ as the time coordinate, we had $d\tau = ds$ along the world line of a galaxy, but when using t_A, $dt_A \neq ds$. Instead, as (23) tells us, $ds = f'\, dt_A$ along the world line of a galaxy.

Thus, the picture we have is that at each space-time point in the ES Universe there are two types of geodesically moving clocks. The cosmological clock records the ephemeris time τ given by the motions of cosmological bodies, while the atomic clock records the atomic time t_A that is measured by vibrations of atoms.

Reconsider now the subspace $\tau = \text{const}$. From (22) it is seen that $d\tau = 0$ means also that $dt_A = 0$, so that a $\tau = \text{const}$ subspace is also a $t_A = \text{const}$ subspace. Thus we have from (23)

$$ds^2(R, t_A = \text{const}) = dR^2 + R^2\, d\Omega^2 \tag{24}$$

which is the same as (12). Therefore, no matter whether we measure time with cosmological clocks recording τ or atomic clocks recording t_A, the proper distance between τ- or t_A-simultaneous events is still measured by the spatial coordinate R. This means that an (R, t_A) space-time diagram will be qualitatively similar to an (R, τ) space-time diagram, with R measuring proper distance between simultaneous events on both diagrams, as shown in Figure 3.

In the following, we are going to maintain R as the spatial coordinate, keeping in mind that it measures proper distance between simultaneous events, whether the simultaneity is determined by τ or t_A. Thus, the basic difference between the approach taken here and the approach of Dirac is

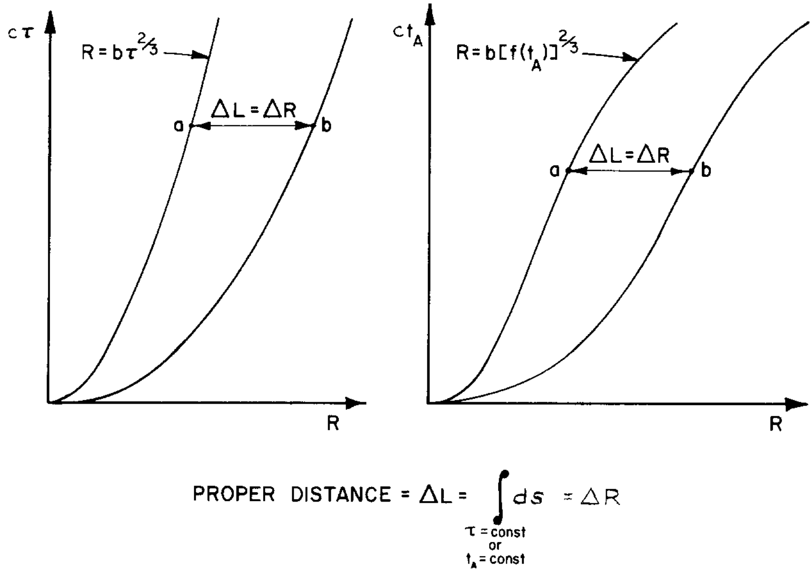

Fig. 3. World lines of galaxies of the ES Universe on (R, τ) and (R, t_A), space-time diagrams. Since τ is a function of only t_A, i.e., $\tau = f(t_A)$, a subspace $\tau = \text{const}$ corresponds to a subspace $t_A = \text{const}$. This means that the proper distance $\Delta L = \Delta R$ between τ-simultaneous events equals the proper distance between corresponding t_A-simultaneous events.

that it is not necessary to introduce an atomic unit of length R_A. Instead, the proper length, which is measured by R, is determined from Einstein's standard theory. In this manner it is not necessary to introduce separate metric forms as Dirac has done.

5. G MEASURED WITH COSMOLOGICAL CLOCKS

The problem at hand is to determine the functional relationship (21) between ephemeris time τ and atomic time t_A. A suggestion for a possible way to proceed will be given in this section by looking at an operational way for measuring G.

The idea of G stems from Newton's second law of motion coupled with Newton's law of universal gravitational attraction, which gives for a test particle of mass m moving in the field of a source of mass M

$$-GM\cancel{m}/R^2 = \cancel{m}d^2R/dt^2 \tag{25}$$

It is found that many exact expressions identical with those from Newtonian theory result when the curvature spatial coordinate R is used in the Einstein field equations. A simple example is that the (invariant) area of sphere is

$4\pi R^2$. The expression for $dR/d\tau$ for a radially moving particle in a Schwarzschild field is $dR/d\tau=(2M/R-2M/R_i)^{1/2}$, which is identical with what is obtained from the Newtonian expression (25) with the Newtonian time t replaced with the proper time τ along the particle's world line. In addition, the curvature invariants for a Schwarzschild field vary as $1/R^n$, where n is a positive integer, indicating an intrinsic singularity at $R=0$.

The ephemeris time τ that determines the evolution of the Universe is measured by the "ticks" of cosmological clocks. As discussed in the previous section, the recording of cosmological ticks is independent of any coordinates we should choose to work with. In turn, the proper distance between τ-simultaneous events, which corresponds to readings made with measuring rods, is also a coordinate-independent quantity. As we have described above, this proper distance is equal to the curvature radial coordinate R. It follows then that quantities constructed from τ and R, such as $v_E=dR/d\tau$ and $a_E=d^2R/d\tau^2$ along the geodesic trajectories of galaxies, will also be coordinate-independent quantities.

Since we know the dynamics of the ES Universe, we can work out v_E and a_E independently of any statements about G. From (20) we know the variation of R with τ for a galaxy is $R=b\tau^{2/3}$, from which we find

$$v_E=dR/d\tau=2b/3\tau^{1/3}=2R/3\tau \tag{26}$$

Substituting the present age of the Universe τ_n into (26) we obtain

$$v_E=HR, \qquad H=\tfrac{2}{3}\tau_n \tag{27}$$

which is the Hubble relationship. From (26) we obtain

$$d^2R/d\tau^2=-2b/9\tau^{4/3}=-2R/9\tau^2 \tag{28}$$

The amount of mass m attracting a galaxy that is a distance R from $R=0$ is found by using the result that in (R,τ) coordinates $\tau_4^4=-\rho(\tau)$ (Gautreau, 1984), so that over a $\tau=\text{const}$ surface

$$M=-4\pi\int \tau_4^4R^2\,dR=4\pi\int \rho R^2\,dR=4\pi\rho\int R^2\,dR=\rho\tfrac{4}{3}\pi R^3 \tag{29}$$

For an ES Universe, $\rho=\frac{1}{6}\pi G\tau^2$ (Gautreau, 1984), so that

$$M=2R^3/9G\tau^2 \tag{30}$$

Substituting the distance to the galaxy, $R=b\tau^{2/3}$, into this, one obtains

$$M=2b^3/9G \tag{31}$$

showing that the mass attracting a galaxy as it evolves is a constant. This is as it should be, because there is zero pressure and the attractive mass is due to the galaxies themselves.

The expressions (26)–(31) have resulted from the ES solution to the standard Einstein field equations. Other than the fact that there is a constant G appearing on the right-hand side of the field equations $G^{\rho\sigma} = -(8\pi G/c^4)T^{\rho\sigma}$, which is there because of appropriate Newtonian flat space-time considerations, we have not yet made any operational identification of G. We now do so. We now define G_E, Newton's gravitational constant measured with ephemeris cosmological clocks, as that quantity for which the motions of galaxies satisfy the relationship

$$-G_E M/R^2 = d^2R/d\tau^2 \tag{32}$$

which is a natural modification of the Newtonian expression (25). When G_E is worked out from (32) by using (28) and (30), one obtains

$$G_E = G \tag{33}$$

Thus, the operational procedure described here gives the result that one wants, namely, a constant value $G_E = G$. If the same procedure were carried out using other coordinates, say, (R, T) or (r, τ) coordinates, the resulting G_E would not be a constant. It is only the measured coordinates (R, τ), whose physical significance we have described above, that yield the desired constant value of G_E with the operational scheme we have desired here.

It seems there is more here than just a coincidence.

6. *G* MEASURED WITH ATOMIC CLOCKS

Just as the "ticks" on cosmological clocks are coordinate independent quantities, so too are the "ticks" on atomic clocks coordinate independent quantities. Thus, although the (thus far undetermined) functional relationship between τ and t_A given by (21) represents a coordinate transformation, it also describes the relationship between the coordinate-independent ticks on the cosmological and atomic clocks that move together on the geodesic world line of a given galaxy.

As we have discussed in Section 4, the curvature radial coordinate R measures proper distance determined by measuring rods between simultaneous events, no matter if the simultaneity is determined by τ or t_A, so that R is a coordinate-independent quantity. Therefore, constructs from the coordinate-independent quantities t_A and R, such as $v_A = dR/dt_A$ and $a_A = d^2R/dt_A^2$ along the geodesic trajectories of galaxies, will be coordinate-independent quantities, albeit having different functional forms from the other coordinate-independent quantities $v_E = dR/d\tau$ and $a_E = d^2R/d\tau^2$.

Essentially what we are discussing here is taking a distance measurement R determined by measuring rods, and finding its rate of change with atomic clocks, just as we did with ephemeris clocks. Thus, $v_E = dR/d\tau$ is the rate of change of R when ephemeris clocks are used, while $v_A = dR/dt_A$ is the rate of change of R when atomic clocks are used.

Let us assume that the functional relationship (21) between τ and t_A is a simple power relation

$$\tau = at_A^m, \qquad a = \text{const} \tag{34}$$

(It is easy to extend the following should the relationship be more complicated.) At our present epoch $\tau = \tau_n$ we have chosen ephemeris and atomic standards such that

$$(d\tau)_n = (dt_A)_n \tag{35}$$

This in turn means that the constant a in (34) must be

$$a = \tau_n^{1-m}/m^m \tag{36}$$

so that

$$\tau = (\tau_n^{1-m}/m^m)t_A^m \tag{37}$$

The distance to a galaxy given by (20) in terms of t_A is

$$R = b\tau^{2/3} = ba^{2/3}t_A^{2m/3} \tag{38}$$

From this we find

$$v_A = dR/dt_A = \tfrac{2}{3}mba^{2/3}t_A^{2m/3-1} \tag{39}$$

and

$$a_A = d^2R/dt_A^2 = \tfrac{2}{3}mba^{2/3}(2m/3-1)t_A^{2m/3-2} \tag{40}$$

The mass M attracting the galaxy will have the same value given in (31), since a $\tau = \text{const}$ integration in (29) will give the same result as the corresponding $t_A = \text{const}$ integration.

In obtaining the above set of expressions, we are still describing motions in the standard ES Universe. The only difference between what we have here and what is described in Section 5 is that we are measuring time with atomic clocks instead of cosmological clocks.

Proceeding in the same spirit as in Section 5, we define G_A, Newton's gravitational "constant" measured with atomic clocks, by setting along the world line of a galaxy

$$-G_AM/R^2 \propto d^2R/dt_A^2 \tag{41}$$

When the above values are substituted into (41) and the proportionality constant adjusted such that $(G_A)_n = G$, we obtain

$$G_A = G(t_A/m\tau_n)^{2m-2} \tag{42a}$$

or equivalently from (37) in terms of τ

$$G_A = G(\tau/\tau_n)^{2-2/m} \tag{42b}$$

7. APPLICATION OF THE LNh

If the dynamical laws governing atomic clocks and the relationship between general relativity and atomic theory were completely known, the functional relationship (21) between τ and t_A could be worked out. Then G_A as defined by (42) would be determined, since $\tau = f(t_A)$ would be known. However, we do not as yet possess a complete theory of atomic processes, much less an understanding of the connection of general relativity with atomic theory, so the functional relationship $\tau = f(t_A)$ cannot be determined from first principles.

The usual approach is to make the assumption that the functional relationship is $\tau = t_A$, from which it would follow that $m = 1$ and $G_A = G_E = G$. It is important to recognize, though, that the statement $\tau = t_A$ is an *assumption* that does not at all follow from the completely separate theories of general relativity and atomic theory. We shall not follow this usual approach of assuming $\tau = t_A$. Instead, we shall invoke Dirac's LNh to determine the functional relationship $\tau = f(t_A)$.

As described in Section 1, the first two of Dirac's large dimensionless numbers require $G \propto t^{-1}$. This is a statement following from Newtonian considerations, with t being Newtonian absolute time. What is now necessary is to incorporate this idea into the formalism of general relativity.

A way of bringing this Newtonian version of the LNh into line with general relativity is to follow Dirac and identify the time t in $G \propto t^{-1}$ with the atomic time t_A used in Section 6. Thus, we will take as our guiding principle the interpretation of the LNh as requiring that when atomic clocks are used for measuring time, as described in Section 6, the variation of G_A as defined by (41) should be

$$G_A \propto t_A^{-1} \tag{43}$$

From (42a) we then find the power m in (37) is $m = 1/2$, resulting in

$$t_A = \tau^2/2\tau_n \tag{44}$$

This is a different relationship between the two times from Dirac's relationship (7). In our proposed scheme for adopting the LNh into general relativity, Dirac's relationship (7) gives $m = 2$, resulting in, from (42a), $G_A \propto t_A^2$. This contradicts the LNh requirement that G_A should vary as t_A^{-1}.

Although our approach gives a time relationship between τ and t_A different from (7) obtained by Dirac, certain expressions that result from our approach turn out to be identical with what Dirac has obtained. For instance, the variation of the distance of a galaxy measured with atomic clocks given by (38) is

$$R = ba^{2/3} t_A^{1/3} \tag{45}$$

which is identical with Dirac's expression (8). This means that the expressions of Dirac that follow from (8), such as the atomic Hubble constant being $H_A = \frac{1}{3} t_A$, will remain unchanged, except for the replacement of Dirac's R_A with Einstein's R.

8. DERIVATION OF LN3

In obtaining (43) and (44) we have used only the first two large numbers (1) and (2). We can therefore *work out* the variation with t_A of the third large number LN3 defined in (3).

Substituting (44) into (30), we obtain the mass up to a distance R at a time t_A as

$$M = R^3/9G\tau_n t_A \tag{46}$$

We will associate the velocity in (3) with v_A, the velocity measured with atomic clocks. To find the distance R_{vA} of the galaxy with a given velocity v_A, we differentiate (45) to get

$$dR/dt_A = v_A = R/3t_A \tag{47}$$

from which

$$R_{v_A} = 3v_A t_A \tag{48}$$

Putting (48) into (46), we find the mass M_{vA} of the galaxies having a speed up to v_A to be

$$M_{v_A} = (3v_A^3/G\tau_n)t_A^2 \tag{49}$$

This holds for any v_A, including Dirac's choice $v_A = c/2$. Since $(\mathrm{LN3})^2 = M_{v_A}/m_p$, we have

$$(\mathrm{LN3})^2 = (3v_A^3/G\tau_n m_p)t_A^2 \tag{50}$$

which agrees with the time variation of the third large number LN3 as given in (5). This means that with the approach given here, LN3 is a consequence of the first two large numbers LN1 and LN2, and so need not be postulated separately.

9. THE NATURAL MICROWAVE RADIATION IN THE LNh

Dirac (1979) has pointed out that the observation of natural microwave radiation, which appears to be of cosmological origin, provides another large dimensionless number that can be incorporated into the scheme of the LNh. The presently observed value of the temperature T_n of the microwave radiation is around 2.8 K, and if this is compared to the rest energy of a proton one obtains

$$KT_n/m_pc^2 = (0.059\times 10^{39})^{-1/3} \tag{51}$$

According to Dirac, in the scheme of the LNh, a dimensionless number which is around $(10^{39})^n$ must vary proportionally to t_A^n, so (51) requires that $T \propto t_A^{-1/3}$.

The following argument, which is similar to one put forth by Dirac (1979), shows that (51) is consistent with the LNh. Assuming the microwave radiation is black body radiation in an expanding universe, each spectral component of the radiation will be red shifted according to the same rule governing the expansion of the galaxies. From (45) the galaxies expand according to $R \propto t_A^{1/3}$, so that

$$\lambda \propto t_A^{1/3} \tag{52}$$

The temperature of the radiation will decrease in the same manner as the frequency f of one of its components, so that

$$T \propto f \propto \lambda^{-1} \propto t_A^{-1/3} \tag{53}$$

which is in agreement with (51).

In obtaining (53) Dirac used an atomic unit of length $R_A \propto t_A^{1/3}$. We see there, though, that the behavior of the natural microwave radiation follows from the LNh when the proper distance to a galaxy, $R \propto t_A^{1/3}$, given by Einstein's standard theory is used as a length measurement.

There is another way to show the agreement between the LNh and the natural microwave radiation. It is well known that the radiation density ρ_r is related to the expansion factor $e^{h(\tau)}$ in (9) by

$$\rho_r \propto [e^{h(\tau)}]^{-4} \tag{54}$$

For an ES Universe, $e^{h(\tau)} \propto \tau^{2/3}$, so that

$$\rho_r \propto \tau^{-8/3} \tag{55}$$

From the Planck law the radiation density ρ_r in terms of mass/volume is

$$\rho_r = \frac{163(KT)^4}{h^3c^5} \tag{56}$$

giving

$$T \propto \tau^{-2/3} \tag{57}$$

Finally, with $\tau \propto t_A^{1/2}$, we have

$$T \propto t_A^{-1/3} \tag{58}$$

which is what is predicted from the LNh.

Using (57), the large number of (51) can be expressed as

$$\text{LN4} = [m_p c^2/KT]^3 = [m_p c^2/KT_n]^3(\tau/\tau_n)^2 \tag{59}$$

The LN4 can be rewritten in terms of densities. Substituting T from (56) into (59), we obtain

$$\text{LN4} = \left[\frac{163 m_p}{\rho_r(h/m_p c)^3}\right]^{3/4} \tag{60}$$

The quantity $h/m_p c$ is the Compton wavelength λ_{Cp} of a proton,

$$\lambda_{\text{Cp}} = h/m_p c \tag{61}$$

If we define the density of a proton, ρ_p, as

$$\rho_p = \frac{m_p}{\frac{4}{3}\pi(\lambda_{\text{Cp}}/2)^3} \tag{62}$$

LN4 in (60) can be written as

$$\text{LN4} = \left[85.3\frac{\rho_p}{\rho_r}\right]^{3/4} = \left[85.3\frac{\rho_p}{\rho_{rn}}\right]^{3/4}\left(\frac{\tau}{\tau_n}\right)^2 \tag{63}$$

where we have used, from (55),

$$\rho_r = \rho_{rn}(\tau_n/\tau)^{8/3} \tag{64}$$

10. OTHER LARGE NUMBERS

There are other large numbers fitting into the scheme of the LNh that apparently have not been considered by Dirac. One such number is

$$\text{LN5} = \frac{\text{distance to the galaxy moving away from us with a speed } v_A}{\text{classical electron radius}}$$

$$= \frac{R_{v_A}}{ke^2/m_e c^2} \approx 10^{39} \tag{65}$$

if $v_A \approx c$. According to the LNh, (65) should vary as t_A. With the velocity in (65) given by v_A, the velocity of a galaxy measured with atomic clocks given in (47), we have from (48)

$$R_{v_A} \propto t_A \tag{66}$$

Thus, LN5 given in (65) varies as t_A, and so in agreement with Dirac's LNh if the relationship between τ and t_A is $t_A \propto \tau^2$.

Another large number is obtained by taking the density of an electron ρ_e obtained from

$$\rho_e = \frac{m_e}{\frac{4}{3}\pi R_e^3}, \qquad R_e = ke^2/m_e c^2 = \text{classical electron radius} \tag{67}$$

and dividing this by the density of matter ρ_m in the Universe as given by the ES relationship

$$\rho_m = \tfrac{1}{6}\pi G\tau^2 \tag{68}$$

to obtain

$$\text{LN6} = \rho_e/\rho_m = \frac{9Gm_e^4c^6}{2(ke^2)^3}\tau^2 \tag{69}$$

With the present age of the Universe $\tau_n \approx 10^{10}$ yr, (69) gives

$$(\text{LN6})_n = (\rho_e/\rho_m)_n \approx 10^{39} \tag{70}$$

From the LNh, one then expects (69) to vary as t_A, which it does with the time relationship $t_A \propto \tau^2$.

11. ALTERNATE DERIVATIONS OF $t_A \propto \tau^2$

In the above, we obtained the relationship $t_A \propto \tau^2$ from the large numbers (1) and (2) together with an operational definition of G_A. We then proceeded to show that the remaining large numbers were in agreement with this time relation.

One can, however, rearrange the logical ordering, and have the relationship $t_A \propto \tau^2$ being a consequence of large numbers other than (1) and (2). To this end, assume an arbitrary functional relationship $\tau = \tau(t_A)$ as in (21), whose functional form is to be determined. With $v_A = dR/dt_A = (dR/d\tau)(d\tau/dt_A)$, the large numbers discussed above then take the form

$$\text{LN1} = \frac{F_C}{F_G} = \frac{ke^2}{Gm_em_p}\left(\frac{t_A}{m\tau_n}\right)^{2-2m} \tag{71}$$

$$\text{LN2} = \frac{t_A}{ke^2/m_ec^3} \tag{72}$$

$$\text{LN3} = \left(\frac{m_{v_A}}{m_p}\right)^{1/2} = \left(\frac{3v_A^3\tau}{4m_p(G(d\tau/dt_A)^3)}\right)^{1/2} \tag{73}$$

$$\mathrm{LN4} = \left(\frac{m_p c^2}{KT}\right)^3 = \left(\frac{m_p c^2}{KT_n}\right)^3 \left(\frac{\tau}{\tau_n}\right)^2 \tag{74}$$

$$\mathrm{LN5} = \frac{R_{v_A}}{R_e} = \frac{\frac{3}{2} v_A}{ke^2/m_e c^2} \frac{\tau}{(d\tau/dt_A)} \tag{75}$$

$$\mathrm{LN6} = \frac{\rho_e}{\rho_m} = \frac{9 G m_e^4 c^6}{2(ke^2)^3} \tau^2 \tag{76}$$

We obtained the relation $t_A \propto \tau^2$ by requiring $\mathrm{LN1} \propto \mathrm{LN2}$ to get $m = \frac{1}{2}$ in (71). However, other combinations of other LN's also yield the same time relationship. For example, requiring $\mathrm{LN2} \propto \mathrm{LN4}$ or $\mathrm{LN2} \propto \mathrm{LN6}$ gives directly $t_A \propto \tau^2$. Similarly, if we require $\mathrm{LN5} \propto \mathrm{LN4}$, we obtain

$$\tau/(d\tau/dt_A) \propto \tau^2 \tag{77}$$

from which it again follows that $t_A \propto \tau^2$.

It can easily be checked that *all* combinations of the LN's are consistent with $t_A \propto \tau^2$. Thus, requiring all the LN's to be proportional to each other provides derivations of the time relationship $t_A \propto \tau^2$ independent from the one given in Section 7.

There is a basic difference between LN2 and the other LN's. LN2 is a statement about two times, the ratio of the age of the Universe to the time for a light signal to traverse an electron. As such, there is no way to derive LN2—it is simply an observational statement. The variations with time of the other LN's, however, are derivable from first principles using Einstein's standard gravitational theory together with the specification that the Universe under consideration shall be the ES Universe.

12. THE TIME VARIATION OF THE LN's

With $t_A = \tau^2/2\tau_n$ we can write the LN's in Section 11 explicitly in terms of t_A and τ once we choose a value for v_A in LN3 and LN5. Dirac took $v_A = c/2$ in LN3 only as a "ball park" value, so there is no fundamental reason for this choice. To relate v_A to a physical quantity, we will choose v_A to be the value of the velocity of the galaxy at the horizon of the observable Universe at the present epoch τ_n.

The cross-hatched regions of Figures 1 and 2 show the observable Universe. Galaxies whose world lines are located in this region have emitted light that has reached our galaxy in the past, and also reaches us at the present epoch τ_n. For example, we presently observe light from the galaxy *aeb* that was emitted at the event *e*.

The furthest galaxy we can present see is galaxy *dh*, whose light reaching us now at our present epoch τ_n was emitted at the big bang at $\tau = 0$. Galaxies further out such as galaxy ij are not able to be seen by us at the present epoch. The horizon galaxy dh has the property that at the present epoch τ_n it intersects the null line $R=3c\tau$, which is the last null line emitted at the big bang that continues in an outward direction (Gautreau, 1984). The intersection event h defines the spatial horizon at the present epoch.

Using the results that the distance from our location to a galaxy is given by $R=b\tau^{2/3}$ and the distance to the spatial horizon is $R_H=3c\tau$, we find the velocity v_{EH} of the galaxy moving through the horizon h, as measured with ephemeris clocks, to be

$$dR/d\tau = v_{EH} = 2c \tag{78}$$

independent of the epoch. Although $v_{EH}>c$, there are no problems with relativistic principles, for the world lines of all galaxies lie within the null cone at all events (Gautreau, 1984).

The velocity v_{AH} of the same galaxy through the horizon h measured with atomic clocks can be obtained from (78) using $v_A=v_E(d\tau/dt_A)$ to get

$$dR/dt_A = v_{AH} = 2c(\tau_n/\tau) \tag{79}$$

so that v_{AH} depends on the epoch. Again, even though it is possible to have $v_A>c$, there are no problems with relativistic principles.

At the present epoch when $\tau=\tau_n$, $(v_{AH})_n=2c$. Putting $v_A=2c$ and $t_A=\tau^2/2\tau_n$ in the LN's of Section 11, we obtain the LN's in terms of atomic time t_A and ephemeris time τ as

$$\mathrm{LN1}=\frac{F_C}{F_G}=\frac{ke^2}{Gm_em_p}\left(\frac{2t_A}{\tau_n}\right)=\frac{ke^2}{Gm_em_p}\left(\frac{\tau}{\tau_n}\right)^2 \tag{80}$$

$$\mathrm{LN2}=\frac{m_ec^3\tau_n}{ke^2}\left(\frac{t_A}{\tau_n}\right)=\frac{m_ec^3\tau_n}{2ke^2}\left(\frac{\tau}{\tau_n}\right)^2 \tag{81}$$

$$\mathrm{LN3}=\left(\frac{M_c}{m_p}\right)^{1/2}=\left(\frac{6c^3\tau_n}{Gm_p}\right)^{1/2}\left(\frac{2t_A}{\tau_n}\right)=\left(\frac{6c^3\tau_n}{Gm_p}\right)^{1/2}\left(\frac{\tau}{\tau_n}\right)^2 \tag{82}$$

$$\mathrm{LN4}=\left(\frac{m_pc^2}{KT}\right)^3=\left(\frac{m_pc^2}{KT_n}\right)^3\left(\frac{2t_A}{\tau_n}\right)=\left(\frac{m_pc^2}{KT_n}\right)^3\left(\frac{\tau}{\tau_n}\right)^2 \tag{83}$$

$$\mathrm{LN5}=\frac{R_c}{R_e}=\frac{3m_ec^3\tau_n}{ke^2}\left(\frac{2t_A}{\tau_n}\right)=\frac{3m_ec^3\tau_n}{ke^2}\left(\frac{\tau}{\tau_n}\right)^2 \tag{84}$$

$$\mathrm{LN6}=\frac{\rho_e}{\rho_m}=\frac{9Gm_e^4c^6\tau_n^2}{2(ke^2)^3}\left(\frac{2t_A}{\tau_n}\right)=\frac{9Gm_e^4c^6\tau_n^2}{2(ke^2)^3}\left(\frac{\tau}{\tau_n}\right)^2 \tag{85}$$

Because we equated v_A to the horizon galaxy's present velocity, $v_A = (v_{AH})_n = 2c$, LN3 and LN5 have the significance that at the present epoch

$$(\mathrm{LN3})_n = \left[\frac{\text{mass in the observable universe}}{\text{mass of a proton}}\right]^{1/2} \tag{86}$$

$$(\mathrm{LN5})_n = \frac{\text{distance to the horizon of the observable universe}}{\text{classical electron radius}} \tag{87}$$

13. THE AGE OF THE UNIVERSE AND THE FUNDAMENTAL CONSTANTS OF NATURE

With $\tau_n = 10^{10}$ yr, $(\mathrm{LN6})_n = 1.2 \times 10^{39}$, which is about half of $(\mathrm{LN1})_n = 2.3 \times 10^{39}$. From (80) and (85) LN6 and LN1 are related by

$$\mathrm{LN6} = \frac{9G^2c^6m_e^5m_p\tau_n^2}{2(ke^2)^4}(\mathrm{LN1}) \tag{88}$$

If we write (88) as

$$\mathrm{LN6} = \tfrac{1}{2}(\tau_n/\tau_f)^2\mathrm{LN1} \tag{89}$$

we see that there is a fundamental time τ_f comparable to the age of the Universe $\tau_n \approx 10^{10}$ yr that can be formed from the fundamental constants of Nature:

$$\tau_f = \frac{(ke^2)^2}{3Gc^3m_e^{5/2}m_p^{1/2}} = 3.05 \times 10^{17}\ \mathrm{sec} = 0.97 \times 10^{10}\ \mathrm{yr} \tag{90}$$

Other groupings of LN's also lead to a similar time.

14. COSMOLOGICAL AND ATOMIC CLOCK PERIODS

From $t_A = \tau^2/2\tau_n$, we can determine the ratio $\beta = d\tau/dt_A$ as

$$\beta(t_A) = (\tau_n/2t_A)^{1/2} \tag{91a}$$

$$\beta(\tau) = \tau_n/\tau \tag{91b}$$

This means that atomic clocks are speeding up with respect to ephemeris clocks. This is just the opposite variation from what follows from Dirac's time relationship (7), which predicts that atomic clocks are slowing down with respect to ephemeris clocks.

To check out (91), one can monitor the ratio β to see if it varies with time (either ephemeris time τ or atomic time t_A). The variation of β if the monitoring is done with atomic clocks is

$$\frac{d\beta/dt_A}{\beta} = -1/2t_A \tag{92a}$$

while if the monitoring is done with cosmological clocks,

$$\frac{d\beta/d\tau}{\beta} = -1/\tau \tag{92b}$$

Equations (92) can be checked with suitable experiments. One possible check could be obtained from the ranging data from the Viking lander on Mars. With proper analysis, this could determine the ephemeris period of Mars relative to atomic clocks on Earth.

15. THE PRINCIPLE OF EQUIVALENCE

It is sometimes stated in the literature that if the ratio β of the periods of a coincident cosmological and atomic clock vary with the age of the Universe, there is then a violation of the principle of equivalence (PE). This, however, is not the case.

The PE can be stated in the following way. Consider a local experiment performed in a reference system in a gravitational field. The PE states that if an identical local experiment is performed in flat space-time in a second reference system that has an acceleration equal to the value of the gravitational field in the first reference system, the results of the two experiments will be identical.

As an example, if the reference system in the gravitational field happens to be in free fall moving along a geodesic, then the equivalent flat space-time reference system must also be moving along a geodesic, i.e., it must be unaccelerated. If, on the other hand, the reference system in the gravitational field is located at some fixed point in a static gravitational field, then the equivalent flat space-time reference system must move along a hyperbolic trajectory of constant acceleration. We have given examples of various local experiments that illustrate the PE in Anderson and Gautreau (1969).

The primary thrust of this statement or most other statements of the PE is that the experiments must be local in nature. Now, the idea of a "local" experiment implies localness in time as well as space. The experiment referred to above of monitoring the periods of a coincident cosmological and atomic clock over a time span of the order of the age of the Universe is not at all local in time. Therefore, one cannot apply the PE to this type of experiment, nor conversely can one use the PE to make statements of the expected outcome of this experiment.

16. DISCUSSION

The starting point of our analysis has been the notion of a cosmological clock measuring ephemeris time τ and an atomic clock measuring atomic time t_A located together at the same point, and therefore following the same geodesic world line through space-time. This world line satisfies the geodesic equations of the standard Einstein theory, and corresponds to the world line of one of the galaxies that is the source of the ES Universe. Proper distance between τ- or t_A-simultaneous events is given by the spatial coordinate R. The trajectory of a galaxy in the ES Universe in the various coordinate systems we have used is given by $R = b\tau^{2/3}$ in (R, τ) coordinates; $R = kt_A^{1/3}$ in (R, t_A) coordinates; $r = \text{const}$ in (r, τ) coordinates.

As far as the author is aware, the operational way of specifying G_A with the LNh as described in Sections 6 and 7 is new. It seems, though, that the application of the LNh described in Section 7 is a most natural extension of the operational result $G_E = G$ established in Section 5.

There are other observations besides the variation of G_A that can be used to establish the time relationship $t_A = \tau^2/2\tau_n$. As has been pointed out in Section 11, there are many combinations of the LN's independent of LN1 involving G_A that can be used to establish the time relationship. We see also that one must be careful when discussing the variation of G, for when determined with ephemeris clocks $G_E = G$, a constant value, while from (42) it is $G_A = G(\tau_n/2t_A) = G(\tau_n/\tau)^2$ that varies with time.

Section 9 shows that the observed natural microwave radiation provides experimental verification of the LNh. Section 14 shows that the formulation of the LNh given in this paper makes a definite prediction about the rates of planetary periods as compared with the rates of atomic periods, which may be possible to detect experimentally.

The existence of different atomic and cosmological times would mean that discussions of elementary particle processes in the early stages of the Universe will have to be reconsidered, since such discussions so far have not taken into account the existence of two separate time evolutions.

COMMENT

Dirac first formulated his LNh in 1938, and believed in its validity until his death in 1984 nearly a half a century later. It seem appropriate, therefore, to dedicate the present work to Dirac, whose personal comments to me on several occasions have sparked my work. Although his methods varied in trying to reconcile the LNh with Einstein's theory of gravitation, whose accomplishments and beauty he greatly admired, Dirac unswervingly maintained the validity of the LNh. In a paper where he considered adopting a generalization of general relativity given by Weyl that involved point-dependent units of length, Dirac (1973) made the following comment about the existence of the large numbers LN1 and LN2:

> It is hard to believe that this is just a coincidence. One suspects that there is some connexion between the two numbers, which will be explained when we have more knowledge of cosmology and of atomic theory.

Subsequently, when he advocated a continuous creation of matter in the Universe to explain the LNh and the variation of G, Dirac (1974) said of the LNh:

> The reason for believing the hypothesis is that without it one does not see how these large numbers could ever be explained.

Dirac (1979) eventually abandoned these approaches to the LNh, and his last version of the LNh involved two metrics in separate Einstein and atomic units.

In the present paper, we have given an alternative way of incorporating the LNh into the standard Einstein theory of gravitation that does not involve two metrics. It thus seems appropriate to close this paper with the following statement made by Dirac (1974):

> The foregoing work is all founded on the Large Numbers hypothesis, in which I have great confidence. It also requires the assumption of two metrics, which is not so certain. The only reason for believing in the two metrics is that up to the present no alternative way of bringing in the Einstein theory has been thought of. But this situation could change.

ACKNOWLEDGMENT

I would like to thank Prof. James L. Anderson for valuable discussions.

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